UNDERSTANDING DECIMALS: THE PATH TO EXPERTISE

Kaye Stacey
University of Melbourne
<k.stacey@edfac.unimelb.edu.au>

Vicki Steinle
University of Melbourne
<v.steinle@edfac.unimelb.edu.au>

This paper reports a longitudinal study of children's understanding of decimal notation. The understandings of 3211 students were classified into four categories and changes over long periods were tracked. The progress of sixty-four students over about three years is reported. When expertise is attained it is generally retained. Over about a year, many students remain in the same misconception category, but in the longer term they move between misconceptions. Improvements and hypotheses to be investigated in the future are noted.

INTRODUCTION AND BACKGROUND

Many students throughout schooling and indeed many adults have difficulty understanding decimal notation. In this paper, we present preliminary results from a longitudinal study that is tracing the development of students' thinking about this challenging topic. As well as being of interest in its own right, this is a case study of how students' understanding develops and/or stays the same with progress through school and in the context of various types of instruction.

There are several ways of classifying the erroneous rules that students may apply when ordering decimals (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985). The coarsest classification is that some students select "longer is larger" (e.g. deciding 0.125 is larger than 0.3) whilst others select "shorter is larger" (e.g. deciding 0.3 is larger than 0.496). Stacey and Steinle (1998), working with interview and written data, traced the various ideas behind these erroneous rules, identified further misconceptions and developed a diagnostic test. This Decimal Comparison Test takes about five minutes and asks students to select the larger decimal from 30 carefully chosen decimal pairs. It enables ten patterns of thinking to be diagnosed. Some of these patterns of thinking are "longer-is-larger" misconceptions, some are "shorter-is-larger" misconceptions and others belong to neither of these. Although future analyses will use the ten patterns of thinking, this paper reports student progress in terms of four major categories (A, L, S and U):

- apparent-experts (A) who may possess excellent understanding or may apply correct rules not understood or may have one identified incorrect pattern of thinkin(Steinle et al (1998)),
- longer-is-larger misconceptions (L) which result from any of five identified patterns of thinking and possibly others,
- shorter-is-larger misconceptions (S) which result from three identified patterns of thinking and possibly others,
- unclassified (U) which is generally a large group since the criteria for classification are quite stringent. It includes students thinking about decimals in unknown ways and others who are inconsistent.

The incidence by age of the various misconceptions about decimal notation is reported in Steinle and Stacey (1998) using cross-sectional data. The longer-is-larger category decreases from Grade 5 (32%) to Year 10 (5%), the trend suggesting that it is unlikely to be common in adult life. The shorter-is-larger category is consistently between 10% and 15% which suggests that this general belief may continue into adulthood. The percentage of apparent-experts also plateaus to about 60% in Year 10, which suggests that there are many adults who have difficulty understanding decimal notation. This paper moves from the cross-sectional analysis to the beginning of a longitudinal analysis, which traces the movement of individuals in the overall data and reports on two questions:

- do students stay in the same category or move frequently from one to the other?
- what are the common paths through the misconceptions to attaining expertise?

Much of this paper provides a similar analysis to that in Stacey and Steinle (submitted), but is based upon a substantially enlarged data set (8708 instead of 5383 tests).

The Longitudinal Sample and Testing

This section presents preliminary results of the longitudinal study from 1995 to 1998. The sample was originally selected to contain a good mix of schools and to maximize the possibility of following students when they changed from primary (Years 0 to 6) to secondary school (Years 7 to 12) at approximately 12 years of age. It consists of classes from:

- one state secondary school in a low socio-economic area and its three "feeder"primary schools,
- one church secondary school in an middle socio-economic area and its main feeder primary school,
- one private girls school in a high socio-economic area with both primary and secondary students,
- two large state primary schools situated in the same middle socio-economic area and the three high schools to which their students mainly progress and
- one church girls' secondary school in a high-middle socio-economic area.

Students were tested with the Decimal Comparison Test at most once every six months, making a total of seven testing times in the data under consideration, from the end of 1995 to the end of 1998. However, schools tested less often than this, for various reasons including different dates for joining the program. The year level distribution of students is shown in Table 1. Note that many students are counted more than once, some up to six times each. In total, 8708 tests are analysed for this paper, although there is no longitudinal data yet for 806 students.

Table 1

Year Level Distribution of Students Completing Tests 1995-1998

Year level	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Total
Number of students	336	965	1468	2114	1917	1067	841	8708

The number of students who have completed the Decimal Comparison Test exactly one, two, three, four, five and six times is shown in Table 2. The students who have completed only one test play no further role in the analysis in this paper, as it aims to track change of individuals from one test to others. For this analysis, the first time an individual undertook the test will be called Test 1, the second time will be called Test 2 and so on. The tests are numbered for the individual, rather than by the date administered. Therefore for some students Test 1 was in 1995 while for others it was in 1996, 1997 or 1998. Because we test intact classes to maximise the usefulness of feedback to teachers, we always have new students entering the testing program. For some students, Tests 1 and 2 have been taken six months apart, whereas for others they may be one year or even 18 months apart if the student was absent on some testing days. This unsatisfactory feature of this preliminary analysis will be addressed in subsequent work.

Table 2
Number of Students by Number of Tests Completed

Exact number of tests completed	One test	Two tests	Three tests	Four tests	Five tests	Six tests
Number of students (N=3211)	806	639	776	718	208	64

RESULTS

Changes of Classification over about Six Months

Table 3 illustrates the changes in classification that occur over consecutive tests. The abbreviations A, L, S and U refer to apparent-expert, longer-is-larger, shorter-is-larger and unclassified groups, respectively. (The classification M is discussed below.) The numbers are amalgamated from all tests. The data is therefore from Test 1 followed by Test 2, from Test 2 followed by Test 3 up to Test 5 followed by Test 6. As noted earlier, these changes are mostly over a period of about six months, but will also include changes over longer periods where students missed out on intermediate testing. This anomaly in the data will be eliminated when the final analysis is done, to give a better measure of change over six months. Another anomaly of the data is that those 1766 students who have been tested more than twice contribute several times to the data. The 64 students who have been tested 6 times, for example, each contribute 5 times to the data in Table 3.

Table 3
Changes in Classification over Consecutive Tests (N = 5497)

Earlier Classification	Later Classification A	L	S	U	M*
A (N= 2851)	2560 (90%)	41 (1%)	69 (2%)	181 (6%)	39 (1%)
L (N= 1179)	295 (25%)	519 (44%)	139 (12%)	226 (19%)	5 (0%)
S (N = 739)	268 (36%)	72 (10%)	265 (36%)	134 (18%)	11 (1%)
U (N = 728)	347 (48%)	78 (11%)	102 (14%)	201 (28%)	18 (2%)
M *(N = 48)	32 (67%)	1 (2%)	3 (6%)	12 (25%)	8 (17%)

* Note M (standing for "misread") is a subset of U (Unclassified) and is therefore shaded.

Table 3 shows that from one test to the next, almost all of the apparent-experts (A) stayed as apparent-experts and approximately one third of other students became apparent-experts. There is a clear tendency for students not becoming experts to retest in the same category. Nearly half of the longer-is-larger (L) students (and nearly two-thirds of those who did not become experts) retested as L. The shorter-is-larger (S) students moved more than the L students, but still about one third stayed in the same category. Amongst the S students who did not become experts, about half remained as S. It will be important to repeat this analysis separating students by age group, as age is likely to be an important determinant of the speed and direction of change.

For interest, Table 3 also shows the numbers of students in the Unclassified category who consistently selected the smaller instead of the larger decimal in the comparison test. These students are labelled M for "misread". They may be A students who have misread the test instructions or they may think that decimal numbers are on the negative side of zero and are therefore ordered in a reverse way. Most of these students retest as A, but the other one third may have a substantial misunderstanding.

Changes of Classification over about Two Years

Table 4 is similar to the Table 3 except that it shows the changes in classification that occurred from Test 1 to Test 4, Test 2 to Test 5 and Test 3 to Test 6. All these students have therefore been tested at least 4 times. Students may have taken these tests only 18 months apart, but since most students missed some testing, for convenience, we have labelled this "changes over about two years". None of the students who were initially in the misread (M) category were tested, although some students have moved into this category.

Table 4
Changes in Classification over about Two Years (N = 1326)

Earlier classification	Later classification A	L	S	U	M
A (N= 505)	471 (93%)	2 (0%)	10 (2%)	22 (4%)	7 (1%)
L (N= 486)	256 (53%)	104 (21%)	35 (7%)	91 (18%)	6 (1%)
S (N = 190)	120 (63%)	8(4%)	38 (20%)	24 (13%)	3(2%)
U (N = 145)	95(66%)	18 (12%)	12 (8%)	20 (13%)	2 (1%)

Over about two years, more than half the students in L, S and U move to A. However, about 20% of students retest in L or S, which probably indicates that school has made little difference to the way they think about decimals, even though in these middle years of school they would be encountered on almost a daily basis.

The fact that 63% of S students move to A, whereas only 53% of L students do, supports the longstanding observation of Sackur-Grivard and Leonard (1985) and others that S is a more sophisticated understanding of decimals than is L. We had previously expected that it would therefore be more likely that L students would become S than vice versa. We had hypothesised that students may move into the more sophisticated misconception on the way to expertise. There is some support for this hypothesis in both Tables 3 and 4, as the percentage of students moving from L to S is greater than the percentage moving from S to L. (It should be noted that the likelihood of movements between L and S are small in comparison with the likelihood of movements to other categories.) The larger number of L students, however, results in a net flow of 67 students from L to S in Table 3, with a corresponding number of 27 in Table 4.

Changes of Classification over about Three Years

A group of 64 students who had reached year 7 (45 students) or Year 8 (19 students) have been tested 6 times, over a period of about three years. Their categories at each of the six tests is shown in Figure 1. How representative is this group of students who have done the test six times? Comparing Figure 1 with cross-sectional results from Steinle and Stacey (1998) discussed earlier, indicates that they are probably better. By Test 6 in Years 7 or 8, there are 48 (75%) apparent-experts compared to about 50% reported previously. There are several reasons why this may be so. Firstly, to be present on the six days of testing indicates that are likely to be regular school attenders with relatively stable schooling. Secondly, the fact that their teachers have made the effort to test six times indicates commitment on their part. This bias will require careful treatment in future analysis.

Figure 1
The Progress of 64 Students over 6 Tests

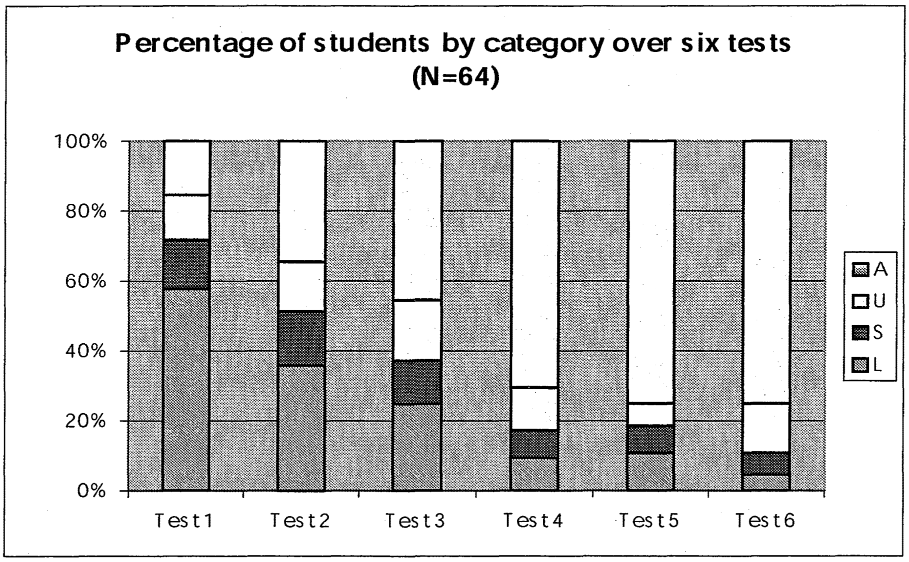


Table 5 shows the initial classification of these 64 students at Test 1 as well as their classification at Test 6. All 10 students who were first tested as A retested as A in Test 6. These students are likely to have been among the most able students (having achieved early expertise in Year 4 or Year 5) so it is not surprising that they test again as experts about three years later. Most of the other students have also become A over the three years, as would be expected. However, 16 students (25%) have not yet reached expertise. There is clearly still work for their Year 8 and 9 teachers to do on fundamental understanding of decimals.

Table 5
Changes in Classification from Test 1 to Test 6 (N = 64)

Earlier classification	Later classification			
	A	L	S	U
A (N= 10)	10 (100%)	0	0	0
L (N= 37)	25 (68%)	2 (5%)	3 (8%)	7 (19%)
S (N = 9)	6 (67%)	1 (11%)	1 (11%)	1 (11%)
U (N = 8)	7 (88%)	0	0	1 (13%)

A First Attempt at Following Individuals to Expertise

This section gives an initial view of the paths that students take through the categories on a series of tests. It is concerned here only with data from the 64 students who completed the test six times (the same data set as Table 5). As noted above, these students entered the study in either Year 4 or 5; their sixth test occurring in Year 7 or 8. Table 5 shows, for example, that of the 37 students who tested as L at Test 1, 25 moved to A at Test 6; 2 retested as L at Test 6, while another 10 moved to S or U at Test 6. This information is displayed as 37 (25,2,10) in the first cell of the second row of Table 6. Thus the first column in Table 6 can be constructed from Table 5. Note that the first cell in the first row of the table, i.e. 10 (10,10,0) indicates that all 10 of the students first tested as A, retested as A in Test 6. In this row, the first two numbers in the brackets must be the same.

Table 6
Numbers of Individuals in each Category at each of the Six Testing Times and, in Brackets, Numbers Indicating Test 6 results. (Sample of students who have been tested 6 times, N=64)

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6
A	10 (10,10,0)	22 (21,21,1)	29 (27,27,2)	45 (42,42,3)	48 (43,43,5)	48
L	37 (25,2,10)	23 (17,2,4)	16 (10,1,5)	6 (2,1,3)	7 (2,1,4)	3
S	9 (6,1,2)	10 (4,3,3)	8 (5,3,0)	5 (1,1,3)	5 (1,2,2)	4
U	8 (7,1,0)	9 (6,1,2)	11 (6,3,2)	8 (3,3,2)	4 (2,1,1)	9

Key: (i,j,k) indicates i students tested as A at Test 6; j students retested at Test 6 in the same category as Test 1, and k students moved to a non-expert category.

Looking along the first row of Table 6 shows that almost all A students retain their knowledge of how to complete the test, although the proportion decreases from Test 1 to Test 5. This probably indicates that students who recently become apparent-experts are the ones most likely to "regress". Those who tested as A at Test 2, for example, have had a long time to consolidate their correct ideas.

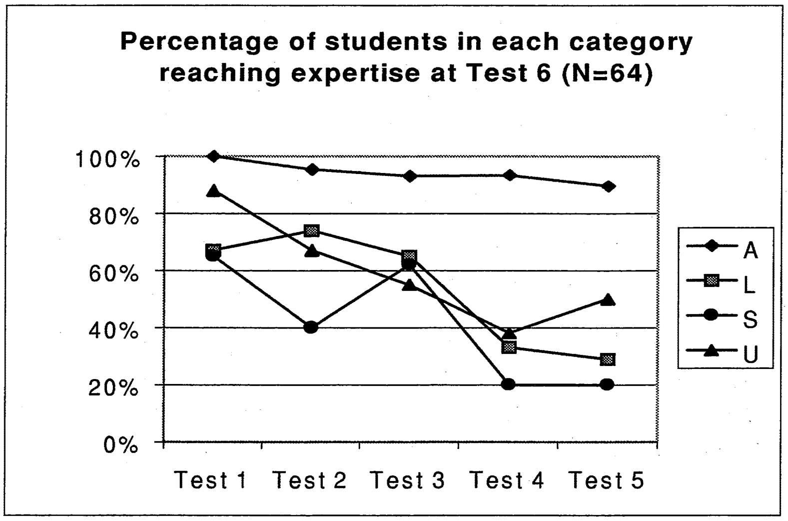
The remaining rows of Table 6 gives information about the movements in and out of the L, S and U categories. By comparing the last number in the brackets (k) in each of these rows, we can see that more students leave the L category and move to a non-expert category, than students in either S or U. So not only to the numbers in the L category decrease, the students who leave this category contribute to the group of students wandering between the S and U categories.

While the numbers of students in the S and U categories seem reasonably stable over the six tests, it should be noted that there is considerable movement in and out of these categories. For example, at Test 2 there are 9 unclassified students, but only 1 of these retests as U in Test 6. Likewise, at Test 4 there are 5 S students; only 1 moves to A in Test 6, another 1 retests as S in Test 6, while 3 students move to either L or U at Test 6. As the number in L is constantly decreasing, it is more likely that these 3 students end up as U at Test 6.

The numbers in S and U stay surprisingly steady, but students move in and out of them over time. This is consistent with our experiences when we interviewed students from a class of Grade 5 and 6 students (not in this sample). These students had recently been studying fractions and so it seemed that the number of shorter-is-larger students who interpreted decimals as fractions (reciprocal thinkers) was inflated by the recent experience.

°As mentioned above, the proportion of students in A who retest in A at Test 6 decreases across the table. Figure 2 illustrates this decrease, as well as a similar decrease in the other categories. Students in a category at a later test are less likely to reach expertise (A) by Test 6 than students who have been in that category earlier.

Figure 2
Percentage of Those Students who Tested in a Given Category at a Given Test who Reach Apparent Expertise at Test 6



DISCUSSION

The purpose of this paper was to report a preliminary analysis of data showing students' progress in their understanding of decimal notation. Several ways in which the analysis can be made more revealing are evident: by following individuals, by following classes so that teaching effects can be observed, by separating the analysis by age group and by using the finer classification system. This analysis is now being planned.

The preliminary analysis has provided the following results to be confirmed later. There is quite marked stability of classification. As we would all hope, there is a general trend towards expertise, and this is showing most clearly in the longest data runs. In the short term those students who do not achieve expertise tend to remain in the same category. After about six months to a year, about a third of students in non-expert categories retested in the same way. Even after a passage of about two years, about 20% of the students with a misconception are classified in the same way. This is an important result given the stringency of the classification criteria and it confirms our informal data (from interviewing previously classified students) that the test, although taking only a few minutes, is highly reliable.

Students in different classifications behave differently. Apparent-experts nearly always stay in this category. This would be expected of students who "really understand" decimals. However, at least in the context of this test, the skill of decimal comparison is well retained even by those who use a rote-learned rule (e.g. compare digits from left to right or add zeros). Those who attain expertise the earliest retain it the best. Following individual paths will help in unravelling students' thinking further and eventually providing better guidelines for teachers.

REFERENCES

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., Peled, I. (1989). Conceptual Bases of Arithmetic Errors: The Case of Decimal Fractions. Journal for Research in Mathematics Education, 20(1), 8-27.

Sackur-Grisvard, C. & Leonard, F. (1985). Intermediate cognitive organization in the process of learning a mathematical concept: The order of positive decimal numbers. Cognition and Instruction, 2, 157-174.

Stacey, K. & Steinle, V. (1998). Refining the Classification of Students' Interpretations of Decimal Notation. Hiroshima Journal of Mathematics Education, 6, 1-21.

Stacey, K. & Steinle, V. (submitted). A Longitudinal Study of children's Thinking About Decimals: A Preliminary Analysis. Proceedings of the Conference of the International Group for the Psychology of Mathematics Education.

Steinle, Vicki & Stacey, Kaye. (1998). The incidence of misconceptions of decimal notation amongst students in Grades 5 to 10. In C. Kanes, M. Goos, E. Warren. (Eds). Teaching Mathematics in New Times, MERGA 21. Vol 2 (pp 548-555) Mathematics Education Research Group of Australasia.

Acknowledgement

This work was carried out with a grant to from the Australian Research Council. We thank the many teachers and students who have participated.